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PASSING THROUGH RESONANCE: THE EXCITATION AND DISSIPATION OF THE LUNAR FREE LIBRATION IN LONGITUDE

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Abstract. The acceleration of the mean lunar longitude has a small effect on the periods of most terms in a Fourier expansion of the longitude. There are several planetary perturbation terms that have small amplitudes, but whose periods are close to the resonant period of the lunar libration in longitude. Some of these terms are moving toward resonance, some are moving away from resonance, and the periods of those terms that do not include the Delaunay variables in their arguments are not moving. Because of its acceleration of longitude, the Moon is receding from the Earth, so the magnitude of the restoring torque that the Earth exerts on the rotating Moon is gradually attenuating; thus resonance itself is moving, but at a much slower rate than the periods of the accelerating planetary perturbations. There are five planetary perturbation terms from the ELP-2000 Ephemeris (with amplitudes of 0″.00001 or greater) that have passed through resonance in the past two million years. One of them is of special interest because it appears to be the excitation source of a supposed free libration in longitude that has been detected by the lunar laser ranging experiment. The amplitude of the term is only 0″.00021 but it could be the source of the $\sim 1''$ amplitude free libration term if the viscoelastic properties of the Moon are similar to those of the Earth.

Key words: Lunar librations, lunar dissipation, lunar Q, free librations

1. The Rigidity of the Moon

The velocity of a seismic shear wave is

$$v_s = \sqrt{\mu/\sigma} \tag{1}$$

where μ is the rigidity and σ is the density of the medium. The structure of the lunar interior is rather homogeneous. Over the depth range 60–1100 km, the shear wave velocity varies little from $v_s = 4.3$ km/s in the model (designated GDT) of Goins, Dainty and Toksoz [1981]; v_s is about five per cent higher over the same depth range in the model (designated N) of Nakamura [1983]. Let M be the mass of the Moon, R its radius, C_{11} its minimum moment of inertia, C_{33} its maximum moment of inertia, and C_{22} the moment of inertia about the third principal axis. Because $\beta = [C_{33} - C_{11}]/C_{22} = 0.000631$ and $C_{33}/MR^2 = 0.392$ [Williams, 1977], all the moments of inertia are close to that of a homogeneous sphere, so the lunar density ($\sigma = 3340$ kg m^{-3}) is fairly uniform. Thus, the lunar rigidity, calculated using (1), does not vary much from $\mu = 62 \times 10^9$ Pa (GDT model) or $\mu = 68 \times 10^9$ Pa (N model). (The lunar rigidity is somewhere between that of cast iron, $\mu = 40 \times 10^9$ Pa, and steel, $\mu = 80 \times 10^9$ Pa.)

For an incompressible non-dissipative homogeneous spherical Moon, its second degree potential-disturbance Love number (omitting the subscript in k_2) is $k = 3/[2 + 19\mu/\sigma g R]$ [Love, pp. 249–259, 1927; Jeffreys, p. 299, 1976], where g is the acceleration of gravity at the lunar surface. Using $\sigma g R = 9.5 \times 10^9$ Pa, the Love

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number determined from seismic data is $k = 0.024$ (GDT model) or $k = 0.022$ (N model). Cheng and Toksoz [1978] used numerical integrations to calculate the Love number for two compressible Moon models in which the elastic properties are functions of depth. The differences between their estimates, $k = 0.029$ and $k = 0.034$, and those from the GDT and N models are principally due their use of earlier shear velocities which are appreciably lower than the more accurate GDT and N velocities.

The Moon deforms elastically, causing the inclination of its equator to the ecliptic to increase by $\delta I = k \times 74''$ [Eckhardt, 1981]. From analyses of lunar laser ranging data, Williams, Newhall and Dickey [1987] estimate that $k = 0.027 \pm 0.006$; that is, the deformation causes the inclination to increase by $2''$ resulting in a -17 m $\sin F$ term (F is the argument of latitude) in the latitude libration. Other libration parameters (e.g., β and C_{33}/MR^2) contribute to the $\sin F$ term in the latitude libration, but they contribute to other terms in different ways, so a thorough analysis allows the deformation term to be discriminated.

To take into account anelastic dissipation in the Moon, the Love number is replaced in the frequency domain by a complex Love number, $k(1 - i/Q)$, where $Q \gg 1$. The term $(1 - i/Q) \approx \exp(-i/Q)$ is actually a phase shifting operator that changes a periodic term of the form $\exp(i\omega t)$ to $\exp[i(\omega t - 1/Q)]$. The imaginary part of the Love number is negative because a deformation lags its inducing forces. The magnitude of the ratio of the real part to the imaginary part of $k(1 - i/Q)$ is Q, where $1/Q$ is the specific dissipation function [Munk and MacDonald, pp. 21-22, 1960] which is frequency dependent. Because $19\mu/\sigma g R \gg 2$, the following approximation is valid for the Moon

$$19\mu k \approx 3\sigma g R. \tag{2}$$

Because $Q^2 \gg 1$, (2) remains valid if k is replaced by $k(1 - i/Q)$ and μ is replaced by $\mu(1 + i/Q)$. The imaginary part of the rigidity is positive because the stress leads the strain [Ben-Menahem and Singh, pp. 856–859, 1981]. A value for k will subsequently be required in order to estimate Q; taking into account the various estimates discussed above, this paper adopts the value $k = 0.025$.

2. Equations Describing the Libration in Longitude of a Deformable Moon

In a selenocentric coordinate system, let $r_i = ru_i$ locate the Earth's center of mass, a distance r from the origin, and let $\rho_i = \rho v_i$ locate a point within the Moon, a distance ρ from the origin. The lunar gravitational potential at r_i is

$$V = (G/r)[\int_{Moon} dM + \sum_{n=2}^{\infty}(R/r)^n \int_{Moon}(\rho/R)^n P_n(u_i v_i)dM]$$

where G is the gravitational constant, P_n is the Legendre function of degree n, and the integrals are over the mass of the Moon. Through degree $n = 2$, the potential

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is

$$V = GM/r + (G/r^3)[\frac{3}{2}\int_{Moon}[(u_i\rho_i)^2 - \rho_i{}^2]dM + \int_{Moon}\rho_i{}^2 dM].$$

Using the moment of inertia tensor,

$$C_{ij} = \int_{Moon}(\rho_k{}^2\delta_{ij} - \rho_i\rho_j)dM,$$

where δ_{ij} is the Kronecker delta, the truncated potential is reformulated as

$$V = GM/r - (G/2r^3)(3u_iC_{ij}u_j - C_{ii}).$$

The potential and moments of inertia of an anelastic Moon are deformed by forces derived from its centrifugal potential, W_1, and from the second-degree Earth-induced tidal potential, W_2. The lunar potential at the Earth is perturbed by

$$\delta V = k(1 - i/Q)(R/r)^5(W_1 + W_2)$$

where

$$W_1 = \frac{1}{2}(\varepsilon_{ijk}\omega_j r_k)^2 = -\frac{1}{2}r_i(\omega_i\omega_j - \delta_{ij}\omega_k{}^2)r_j,$$

and

$$W_2 = (Gm/2r^3)r_i(3u_iu_j - \delta_{ij})r_j,$$

where ω_i is the lunar angular rotation rate, m is the mass of the Earth, and ε_{ijk} is the alternating tensor. Then

$$\delta V = -(G/2r^3)(3u_i\delta C_{ij}u_j - \delta C_{kk}\delta_{ij}) + k(1 - i/Q)R^5\omega_k{}^2/3r^3$$

where the perturbation of the moment of inertia tensor is

$$\begin{aligned}\delta C_{ij} &= (1 - i/Q)K[\omega_i\omega_j - 3(Gm/a^3)(a/r)^3u_iu_j] \\ &= (1 - i/Q)K[\omega_i\omega_j - 3n^2(a/r)^3u_iu_j] \\ &\approx (1 - i/Q)K(\omega_i\omega_j - 3n^2u_iu_j),\end{aligned} \tag{3}$$

a is the mean Earth-Moon distance,

$$K = kR^5/3G = \frac{k(MR^2)(R/a)^3}{(M/m)(3n^2)}, \tag{4}$$

and n is the mean rate of lunar motion.

For a small lunar rotation, $\delta\varphi_k$, the Earth's direction cosines change by $\delta u_i = \varepsilon_{ijk}u_j\delta\varphi_k$, so $\partial u_i/\partial\varphi_k = \varepsilon_{ijk}u_j$. The torque exerted on the Moon is

$$N_k = m\partial(V + \delta V)/\partial\varphi_k = (3Gm/r^3)u_i(C_{ij} + \delta C_{ij})\varepsilon_{jlk}u_l. \tag{5}$$

In an inertial frame, the time derivative of the angular momentum, $L_i + \delta L_i = (C_{ij} + \delta C_{ij})\omega_j$, is equal to the torque, so

$$d(L_i + \delta L_i)/dt = \dot{L}_i + \delta \dot{L}_i + \varepsilon_{ijk}\omega_j(L_k + \delta L_k) = N_i, \tag{6}$$

where the dot derivative is with respect to time in the lunar coordinate system which is chosen to be the same as the principal axes of C_{ij}. Setting (5) equal to (6) with $\delta C_{ij} = 0$ gives the basic Euler dynamical equations for the rotation of a rigid Moon.

If the Love number is real, the centrifugal potential deforms the Moon as an oblate spheroid whose shortest axis is the instantaneous rotation axis, so W_1 does not affect the rotation rate through the RHS of (6), only its direction. Also if the Love number is real, the tidal potential deforms the Moon as a prolate spheroid with its longest axis in the instantaneous direction of the Earth, so there can be no torque due to W_2. The real part of the Love number only has an effect on the lunar rotation rate through the change in the Moon's moment of inertia, and that is almost entirely due to the centrifugal potential. The mean lunar rotation rate n (the same as its mean rate of motion) is much larger than its perturbation τ, the libration in longitude, so the real part of the Love number affects the Moon's librations principally through a "frozen in" deformation in its moment of inertia tensor which is a permanent part of C_{ij}. The impact of the real part of the Love number on the rotation rate is essentially nil; only the imaginary part contributes significantly to perturbations from the Euler dynamical equations. This can be demonstrated mathematically by combining (3), (4), (5) and (6).

For examining the libration in longitude τ (about the 3-axis), use the approximations

$$\omega_i = \begin{bmatrix} 0 \\ 0 \\ \omega_3 \end{bmatrix}, u_i = \begin{bmatrix} 1 \\ u_2 \\ 0 \end{bmatrix},$$

$$\omega_3 = n + \dot{\tau} \quad \& \quad u_2 = s - \tau,$$

where the 1-axis is in the mean Earth direction (or nearly so [Eckhardt, 1973]) and s is comprised of the equation of the center and all inequalities in longitude. With $\gamma = (C_{22} - C_{11})/C_{33} = 0.000228$ and, approximately, $\omega_0{}^2 = 3\gamma n^2$, the bottom row of the Euler equation becomes approximately

$$\ddot{\tau} = \omega_0{}^2(s - \tau). \tag{7}$$

Excitation and dissipation require hysteresis, so only the imaginary periodic terms of the moment of inertia perturbation,

$$\delta C_{ij} = (1 - i/Q)K \begin{bmatrix} -3n^2 & -3n^2 u_2 & 0 \\ -3n^2 u_2 & -3n^2 u_2{}^2 & 0 \\ 0 & 0 & \omega_3{}^2 \end{bmatrix},$$

are significant. Therefore, to the first order in s, τ and $\dot{\tau}$, the significant portion of δC_{ij} is

$$\delta\tilde{C}_{ij} = -i(K/Q)\begin{bmatrix} 0 & -3n^2(s-\tau) & 0 \\ -3n^2(s-\tau) & 0 & 0 \\ 0 & 0 & 2n\dot{\tau} \end{bmatrix}.$$

Dropping second order terms, the variable perturbation in the lunar angular momentum is

$$\delta\tilde{L}_i = \delta\tilde{C}_{ij}\omega_j = -2i(K/Q)n^2\dot{\tau}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

and its rate of change is

$$\delta\dot{\tilde{L}}_i = \delta\dot{\tilde{C}}_{ij}\omega_j = -2i(K/Q)n^2\ddot{\tau}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Then

$$\delta\tilde{C}_{ij}u_j = 3i(K/Q)n^2\begin{bmatrix} 0 \\ s-\tau \\ 0 \end{bmatrix}$$

and

$$\varepsilon_{kli}u_l\delta\tilde{C}_{ij}u_j = 3i(K/Q)n^2\begin{bmatrix} 0 \\ 0 \\ s-\tau \end{bmatrix}.$$

The perturbation of the torque exerted on the deformed Moon by the Earth is then

$$\delta\tilde{N}_i = m\varepsilon_{ijk}u_j[\partial(\delta\tilde{V})/\partial u_k] = 9i(K/Q)n^4\begin{bmatrix} 0 \\ 0 \\ s-\tau \end{bmatrix}.$$

The perturbation of (7) accounting for the deformable Moon model is

$$(1 - 2i\gamma\kappa/3)\ddot{\tau} = (1 + i\kappa){\omega_0}^2(s-\tau) \tag{8}$$

where $\kappa = \varsigma k/Q$ and $\varsigma = (R/a)^3(MR^2/C_{33})\gamma^{-1}(m/M) = 0.0841$. Take $k = 0.025$ and suppose that $Q > 20$ so that $\kappa < 10^{-4}$. Because $2\gamma \ll 3$, the $2i\gamma\kappa/3$ term is dropped, and (8) becomes

$$\ddot{\tau} = {\omega_0}^2(1 + i\kappa)(s-\tau). \tag{9}$$

The period of resonance, $2\pi/\omega_0$, derived by detailed semi-analytic theories has a small uncertainty. Estimates based on the LURE 2 constants [Williams, 1977] are, in sidereal months, 38.651 [Eckhardt, 1981], 38.654 [Moons, 1982] , and 38.658 [Migus, 1980]. These estimates are adjusted for the JPL constants of Williams, Newhall and Dickey [1987]. The increase in γ is $(227.951 - 227.370) \times 10^{-6} = 0.581 \times 10^{-6}$ which decreases each estimate by 0.049, but using the technique of Williams *et al.* [1973], the changes in the harmonic coefficients C_{31} and C_{33} work out to be the same as decreasing γ by 0.592×10^{-6} and the adjustments almost cancel each other. The net result is that each estimate increases by only 0.001; this reflects the fact that the lunar laser ranging experiment is more sensitive to the period of τ resonance than to the related solution parameters.

3. Analytic Solutions of the Libration Equations

Let the longitude source terms be

$$s = \sum s_k = \sum H_k \exp(i\omega_k t)$$

where each H_k is a complex coefficient. Then the solutio . .. (9) is

$$\tau = \sum \tau_k = \sum \frac{(1 + i\kappa)s_k}{1 + i\kappa - (\omega_k/\omega_0)^2} \approx \sum \frac{s_k}{1 - (\omega_k/\omega_0)^2}. \tag{10}$$

The $i\kappa$ term has negligible effect on the steady state solution given by (10) because currently none of the s_k source terms has a frequency close enough to resonance. If, however, the argument of some source term includes an acceleration term,

$$s_k = H_k \exp[i(\omega_k t + \frac{1}{2}c_k t^2)], \tag{11}$$

the dissipation term may be important, especially near resonance. Perturbing this process is the fact that because ω_0 is proportional to n, it is a linear function of time (because of the acceleration of the mean longitude of the Moon).

Suppose that there is a term of the form (11) that carried τ through resonance in the recent past (using a geological sense of time so that a million years or so ago was recent) at $t = 0$ so that $\omega_k(0) = \omega_0(0)$. Change the time variable to the dimensionless variable η where $\dot{\eta} = \omega_0(0)$ so that η goes through 2π radians every 38.65 sidereal months, and use the prime ($'$) to denote the derivative with respect to η. The argument of s_k in (11) is then expressed in the form

$$\omega_k(0)t + \frac{1}{2}c_k t^2 = \omega_0(0)t + \frac{1}{2}c_k t^2 = \eta + \frac{1}{2}\lambda_k \eta^2,$$

in which case

$$\lambda_k = c_k/{\omega_0}^2(0).$$

In the Fourier expansion of the longitude of the Moon, many of the planetary source terms near resonance have accelerations with approximately the same magnitude as the acceleration of the mean longitude of the Moon [Williams, Newhall and Dickey, 1992],

$$\dot{n} = (-26.0 \pm 1.0)''/\text{cy}$$

or

$$\dot{n}/n = -0.150 \times 10^{-9}/\text{y}.$$

These are the only planetary terms of any significance for this study. Thus, with $\omega_0(0) = 2.17/\text{y}$,

$$|\lambda_k| \approx 26''.0\,\text{cy}^{-2}/\omega_0{}^2(0) = 2.67 \times 10^{-9}. \quad (12)$$

Let the resonance angular rate as a function of η be represented by $\nu(\eta)$; with this choice, $\nu(0) = 1$. Also define $\lambda_0 = \nu'$. Because ν is proportional to n,

$$\frac{\dot{n}}{n} = \frac{d\ln n(t)}{dt} = \omega_0(0)\frac{d\ln \nu(\eta)}{d\eta} = \omega_0(0)\lambda_0,$$

and, therefore,

$$\lambda_0 = -0.150 \times 10^{-9}\text{y}^{-1}/\omega_0(0) = -0.069 \times 10^{-9}. \quad (13)$$

Then, on dropping the negligible η^2 term,

$$\omega_0{}^2(t) = \frac{\omega_0{}^2(0)n^2(t)}{n^2(0)} = \omega_0{}^2(0)[1 + 2\frac{\dot{n}}{n(0)}t] = \omega_0{}^2(0)(1 + 2\lambda_0\eta).$$

Change the independent variable of (9) from t to η,

$$\tau_k'' + (1 + 2\lambda_0\eta)(1 + i\kappa)\tau_k = (1 + 2\lambda_0\eta)(1 + i\kappa)H_k \exp[i(\eta + \lambda_k\eta^2/2)]. \quad (14)$$

Next substitute

$$\tau_k(\eta) = \xi_k(\eta)\exp(i\eta) \quad (15)$$

into (14), where $\xi_k(\eta)$ is a function that varies slowly compared with $\exp(i\eta)$ so that $|\xi_k''(\eta)| \ll |\xi_k'(\eta)|$. Then, on neglecting ξ_k'' and all terms quadratic in κ, λ_0, H_k,

$$\xi_k' + (\kappa/2 - i\lambda_0\eta)\xi_k = -i(H_k/2)\exp(i\lambda_k\eta^2/2). \quad (16)$$

Using the integrating factor $\exp[(\kappa\eta - i\lambda_0\eta^2)/2]$, the solution to (16) at $\eta = h$ is

$$\xi_k(h) = -\frac{i}{2}H_k \exp[\frac{-\kappa h + i\lambda_0 h^2}{2}] \int_{-\infty}^{h} \exp[\frac{i(\lambda_k - \lambda_0)\eta^2 + \kappa\eta}{2}]d\eta. \quad (17)$$

From (12) and (13), it is clear that λ_k dominates the $\lambda_k - \lambda_0$ term. The integral for $\lambda_0 > \lambda_k$ (λ_k obviously being negative) is

$$\int_{-\infty}^{h} \exp[\frac{-i(\lambda_0 - \lambda_k)\eta^2 + \kappa\eta}{2}]d\eta = \exp[-i(\kappa/p_k)^2/8] \int_{-\infty}^{h} \exp(-w^2)d\eta \quad (18)$$

where

$$w = w(\eta) = (1 + i)p_k\eta/2 - (1 - i)\kappa/(4p_k) \tag{19}$$

and

$$p_k = |\lambda_0 - \lambda_k|^{1/2}. \tag{20}$$

(The absolute value is used in this definition so that it will apply as well when λ_k is positive.) Substituting $(1 - i)p_k{}^{-1}dw = d\eta$,

$$\int_{-\infty}^{h} \exp(-w^2)d\eta = \frac{1-i}{p_k}\int_{-\infty}^{w(h)} \exp(-w^2)dw = \frac{1-i}{2p_k}\sqrt{\pi}[1 + \mathrm{erf}[w(h)]], \tag{21}$$

where $\mathrm{erf}[w(h)]$ is the error function with the complex argument $w(h)$.

The phase of H_k is the phase of s_k precisely at $t = 0$. There is no way of knowing the phase of H_k, so there is no way of knowing the phase of τ_k near resonance. Only the magnitude of τ_k can be calculated as a function of $|H_k|$, p_k, κ and h; combining (15), (17), (18) and (21), it is

$$|\tau_k(h)| = \sqrt{\pi/2}[|H_k|/(2p_k)]\exp(-\kappa h/2)|1 + \mathrm{erf}[w(h)]|. \tag{22}$$

The integral (18) for $\lambda_k < \lambda_0$ is the complex conjugate of the integral for $\lambda_k > \lambda_0$, so (22) is valid for all values of $\lambda_0 - \lambda_k$ except, of course, for (the non-existent) $\lambda_k = \lambda_0$ which represents being parked at resonance.

If $2p_k{}^2h \gg \kappa$,

$$w(h) \approx (1 + i)p_k h/2 \tag{23}$$

and

$$\frac{1-i}{2p_k}\sqrt{\pi}[1 + \mathrm{erf}[w(h)]] \approx \frac{\sqrt{\pi}}{2p_k}[[1 + 2\mathrm{C}(p_k h/\sqrt{\pi})] - i[1 + 2\mathrm{S}(p_k h/\sqrt{\pi})]],$$

where $\mathrm{C}(p_k h/\sqrt{\pi})$ and $\mathrm{S}(p_k h/\sqrt{\pi})$ are the Fresnel integrals. Then

$$|\tau_k(h)| \approx \sqrt{\pi}[\frac{|H_k|}{4p_k}]\exp(-\frac{\kappa h}{2})\sqrt{[1 + 2\mathrm{C}(\frac{p_k h}{\sqrt{\pi}})]^2 + [1 + 2\mathrm{S}(\frac{p_k h}{\sqrt{\pi}})]^2}. \tag{24}$$

For $p_k h \gg \sqrt{\pi}$, $\mathrm{C}(p_k h/\sqrt{\pi}) \approx \mathrm{S}(p_k h/\sqrt{\pi}) \approx 1/2$, so (24) becomes

$$|\tau_k(h)| \approx \sqrt{\pi/2}[|H_k|/p_k]\exp(-\kappa h/2). \tag{25}$$

4. The Accelerating Arguments

Let $m = n'/n = 0.0748$ *currently* where n' is the mean rate of solar motion. Numerical integrations of the orbits of the Solar System planets demonstrate that their mean motions have inappreciable accelerations over millions of years [Quinn, Tremaine and Duncan, 1991], so $\dot{n}' = 0$ and $\dot{m}/m = -\dot{n}/n$. Using the m power series for the mean motions of the lunar node Ω and perigee $\tilde{\omega}$ [Brouwer and Clemence, pp. 322-323, 1961], their accelerations are then related to $\dot{n}$ by

$$\ddot{\Omega} = (1.5m^2 - 0.84375m^3 - 8.531252m^4 - 23.91845m^5 \\ -48.6504m^6 - 79.016m^7)\dot{n} = 0.0077\dot{n}$$

and

$$\ddot{\tilde{\omega}} = (-1.5m^2 - 21.03125m^3 - 127.218752m^4 - 648.17625m^5 \\ -3130.5256m^6 - 15118.887m^7 - 80282.32m^8 \\ -42578.3m^9)\dot{n} = -0.0236\dot{n}.$$

The accelerations of all components of the planetary perturbation arguments are essentially zero except for the Delaunay variables D, l and F which have the accelerations, respectively, $\dot{n}$, $\dot{n} - \ddot{\tilde{\omega}} = 1.0236\dot{n}$, and $\dot{n} - \ddot{\Omega} = 0.9923\dot{n}$; in units of η, they are $\dot{n}/\omega_0{}^2(0)$, $1.0236\dot{n}/\omega_0{}^2(0)$, and $0.9923\dot{n}/\omega_0{}^2(0)$ where $\dot{n}/\omega_0{}^2(0) = -2.67 \times 10^{-9}$. For a Fourier term in the planetary longitude perturbation of the Moon to be of interest, its period must be near resonance and, with its argument expressed as an increasing function of time, the sum of the integer coefficients of D, l and F in the argument must be equal to (or greater than) 1 if the term's period is greater than the resonant period or the sum must be equal to (or less than) -1 if its period is less than the resonant period. The effect of the slowly changing resonance (the λ_0 term) can be overlooked in this winnowing compared with the uncertainty in the period of resonance, a value that is theory and parameter dependent. Source terms from the Chapront and Chapront-Touzé [1983] ELP-2000 lunar ephemeris that meet these criteria are given in Table I. V, T and M represent the mean longitudes of Venus, Earth and Mars.

The nominal period of resonance is chosen as 38.652 sidereal months [Eckhardt, 1981]; adjustments for the resonant periods of Migus [1980] and Moons [1982] will be considered later. Because of their amplitudes and closeness to resonance (one above and the other below), the last two terms in Table I are of principal interest. The other three are insignificant and will be neglected. For the subscripts designating these terms, use 1 for 16331 and 2 for 16018.

Term 16331 is closest to resonance and, by far, the most important of all terms that have "recently" passed through resonance. Its acceleration is $(2 - 1.0236)\dot{n}/\omega_0{}^2(0)$, so

$$\lambda_1 = 0.9764 \times (-2.67 \times 10^{-9}) = -2.61 \times 10^{-9},$$

TABLE I

Planetary perturbations in longitude for terms that have "recently" passed through resonance and whose periods differ from the resonance period by less than one per cent. These terms were selected from the ELP-2000 lunar ephemeris of Chapront and Chapront-Touzé [1983]. The sequence numers are those of ELP-2000

Sequence Number	$\|H_k\|$ (Arc-Seconds)	Argument (Excluding Phase)	Period (Sidereal Months)
17353	0.00011	$-39V + 39T + 2D$	39.080
16766	0.00003	$-24V + 30T - 4M + 2D - l$	39.039
16979	0.00001	$-26V + 29T + 4M + 2D - l$	38.765
16331	0.00021	$-21V + 23T + 2D - l$	38.666
16018	0.00066	$19V - 18T - D + l - F$	38.258

and the λ_0 term just marginally contributes to making [see (13) and (20)]

$$p_1{}^2 = 2.54 \times 10^{-9}.$$

Then, because the dimensionless angular rate at resonance is unity,

$$1 + p_1{}^2 h = 38.666/38.652 = 1 + 0.000362$$

and

$$h = 143000.$$

Resonance occurred $(2.8912/2\pi)h = 66000$ years ago. At that time, its libration amplitude was half of

$$\sqrt{\pi/2}|H_1|/p_1 = 5''.2$$

and, assuming that (25) affords a close enough approximation, its amplitude is now

$$|\tau_1(h)| \approx 5''.2 \exp(-71000\kappa). \tag{26}$$

The more precise equation (22) will be applied later. Following the same calculations for Term 16018 gives

$$|\tau_2(h)| \approx 16''.1 \exp(-1.9 \times 10^6 \kappa). \tag{27}$$

From the ratio (26)/(27),

$$|\tau_1(h)|/|\tau_2(h)| \geq 1 \text{ for } \kappa \geq 0.62 \times 10^{-6}. \tag{28}$$

From the analysis of a six year span of lunar laser ranging data, Calame [1977] estimated the free libration in longitude to have a $1''.8$ amplitude. She did not,

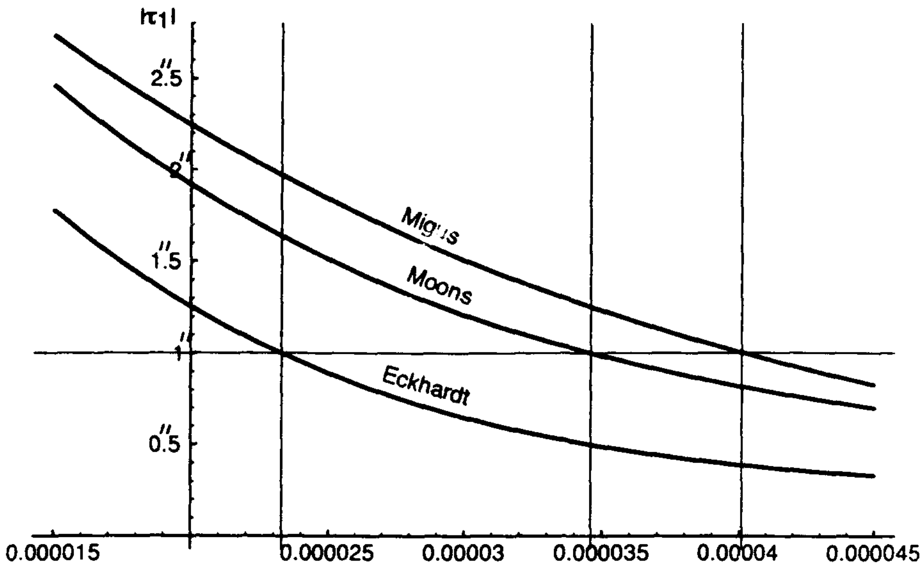

Fig. 1. Plots of $|\tau_1|$ at the current epoch as functions of κ. For the Eckhardt, Moons and Migus resonances, the solutions to $|\tau_1| = 1$ are indicated at $\kappa = 23.2 \times 10^{-6}$, $\kappa = 34.6 \times 10^{-6}$, and $\kappa = 40.1 \times 10^{-6}$.

however, allow for the planetary terms in lunar longitude. Except for $\tau_1(h)$ and $\tau_2(h)$, the neglected terms (principally ELP-2000 Term 6649 which has amplitude $0''.85$, argument $3T - 5M$ and period 39.138 sidereal months [Eckhardt, 1982]) account for only about half of her $1''.8$ estimate. Suppose that the approximately $1''.0$ residual is due to $\tau_1(h)$ or $\tau_2(h)$. If it were entirely due to $\tau_1(h)$ then, by (26),

$$\kappa = 23 \times 10^{-6}, \tag{29}$$

and if it were entirely due to $\tau_2(h)$ then, by (27), $\kappa = 1.5 \times 10^{-6}$; but, by (28), $|\tau_1(h)| > |\tau_2(h)|$ if $\kappa = 1.5 \times 10^{-6}$, so (29) provides the single estimate of the dissipation parameter for the Eckhardt [1981] resonance and $1''.0$ residual. This is only an approximate solution and modifications are also required for the Moons [1982] and Migus [1980] resonances. Moreover the $1''.0$ residual is only a nominal value. A plot of $|\tau_1(143000)|$ [Eckhardt], $|\tau_1(112000)|$ [Moons], and $|\tau_1(71000)|$ [Migus] as functions of κ by the general solution (22) is presented in Figure 1. The solutions to $|\tau_1(h)| = 1$ are indicated at

$$\kappa = 23.3 \times 10^{-6} \quad \text{[Eckhardt]}, \tag{30}$$

$$\kappa = 34.6 \times 10^{-6} \quad \text{[Moons]},$$

and

$$\kappa = 40.1 \times 10^{-6} \quad \text{[Migus]}.$$

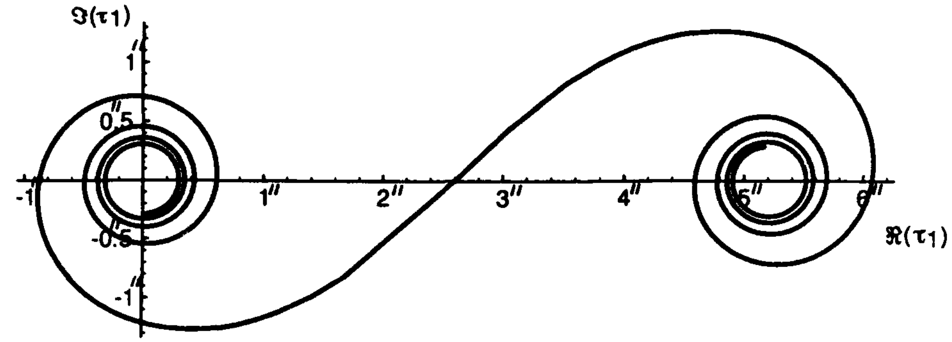

Fig. 2. The solution to (25) for ELP-2000 Term 16331 from 132000 years ago until the present. Resonance according to the adopted model occurred 66000 years ago. The coordinate units are arc-seconds. There exists no free libration with an amplitude as large as 5″ as shown in this solution, so the effect of dissipation must be included to get a plausible model.

Although Moons' resonance is closer to Eckhardt's than Migus', her solution is closer to Migus' than Eckhardt's at the current epoch. To understand how this can be, and to gain insight into the nature of the solutions, the solution paths in the complex plane are examined.

5. The Solution Paths

If $\kappa = 0$ and $\lambda_0 = 0$, Equation (16) becomes

$$\xi_k' = -i(H_k/2)\exp(i\lambda_k\eta^2/2). \tag{31}$$

Because $|\xi_k'| = |H_k/2|$, $\xi_k(\eta)$ follows a path in the complex plane at a constant rate with respect to η. The direction of the path is given by the the sum of the phase of $-iH_k$ and the argument of the exponential term. Suppose that $\lambda_k < 0$; then for $\eta < 0$, the path is clockwise, and for $\eta > 0$, the path is counter-clockwise. The angular rate is proportional to η^2 so, for $\eta < 0$, the rotation rate decreases, and for $\eta > 0$, the rotation rate increases. The net result if $-iH_k$ is real and positive is that the solution follows the Cornu spiral [Born and Wolf, pp. 430-433, 1970] depicted in Figure 2. (The rotations are in the opposite directions for $\lambda_k > 0$ and the orientation of the spiral depends on the phase of $-iH_k$.)

If $\kappa = 0$ and $H_k = 0$, Equation (16) becomes

$$\xi_k' = i\lambda_0\eta\xi_k. \tag{32}$$

In this case, $\xi_k = |\xi_k|\exp(i\lambda_0\eta^2/2)$ follows a circular path in the complex plane with an angular rate proportional to $|\lambda_0\eta|$. The path is counter-clockwise for $\lambda_0 < 0$, and it is clockwise for $\lambda_0 > 0$. A combination of (31) and (32) results in a Cornu

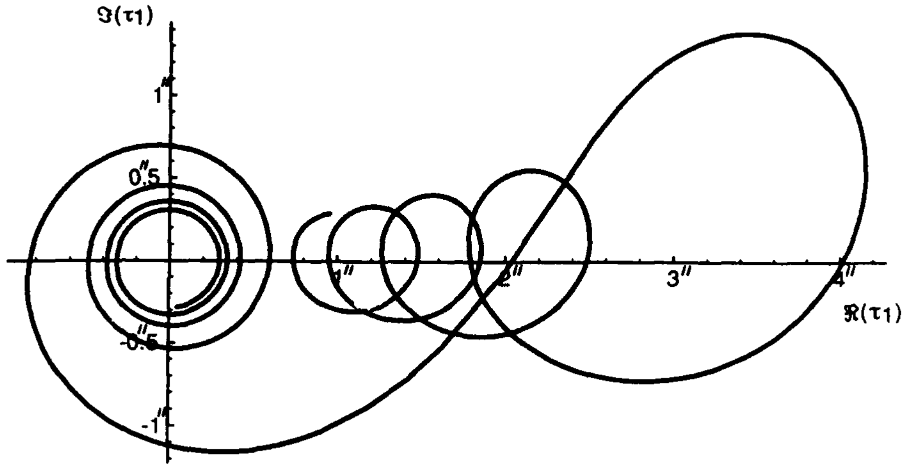

Fig. 3. The solution to (20) for ELP-2000 Term 16331 with the same units and over the same time span as Figure 2. The model is modified by including a dissipation term. The current 1'' amplitude of $\tau_1(143000)$ shown in this plot would be perceived as a free libration.

spiral with the same dimensions as as (31); the only effect of (32) is to rotate the Cornu spiral without changing its dimensions. For the current problem, (32) is only of marginal interest.

If $\lambda_0 = 0$ and $\lambda_k = 0$, Equation (16) becomes

$$\xi_k' = -(\kappa/2)\xi_k, \tag{33}$$

so $\xi_k(\eta)$ follows an exponential decay path in the complex plane that is a straight line directed toward the origin. In the general solution of (16), $\xi_k(\eta)$ both spirals as in (31) and decays as in (33). Well before passing through resonance, when $p_k\eta \ll -1$, $\xi_k(\eta)$ moves in a tight spiral around the origin and the decay term has little effect. As $p_k\eta$ becomes small and $\xi_k(\eta)$ moves through the region of the resonance inflection point, (31) dominates (33). Eventually, when $p_k\eta \gg 1$, $\xi_k(\eta)$ moves in an increasingly tight spiral around a focus that decays toward the origin, so (33) dominates (31). The general solution to (16) is given by (22) which can conveniently be evaluated and plotted using Mathematica [Wolfram, 1988]. A plot of the the $\tau(\eta)$ solution path according to (22) (retaining the relative phase from the error function) is shown in Figure 3. The dissipation parameter used is given by (30).

For the shorter time intervals and higher dissipations that are demanded by the Moons and Migus resonances to attain a 1''.0 amplitude for $\tau_1(h)$, their solution curves, shown in Figure 4, are markedly different. The two curves in Figure 4 are currently headed in opposite directions. These directions are relatively insensitive to either κ used. As time increases, the $|\tau_1|$ in the top plot is decreasing and the $|\tau_1|$ in the bottom plot is increasing. Thus the entire curve for the Moons solution ($|\tau_1| = 1''.0$ at $\kappa = 34.6 \times 10^{-6}$) is moving downward with time whereas the entire

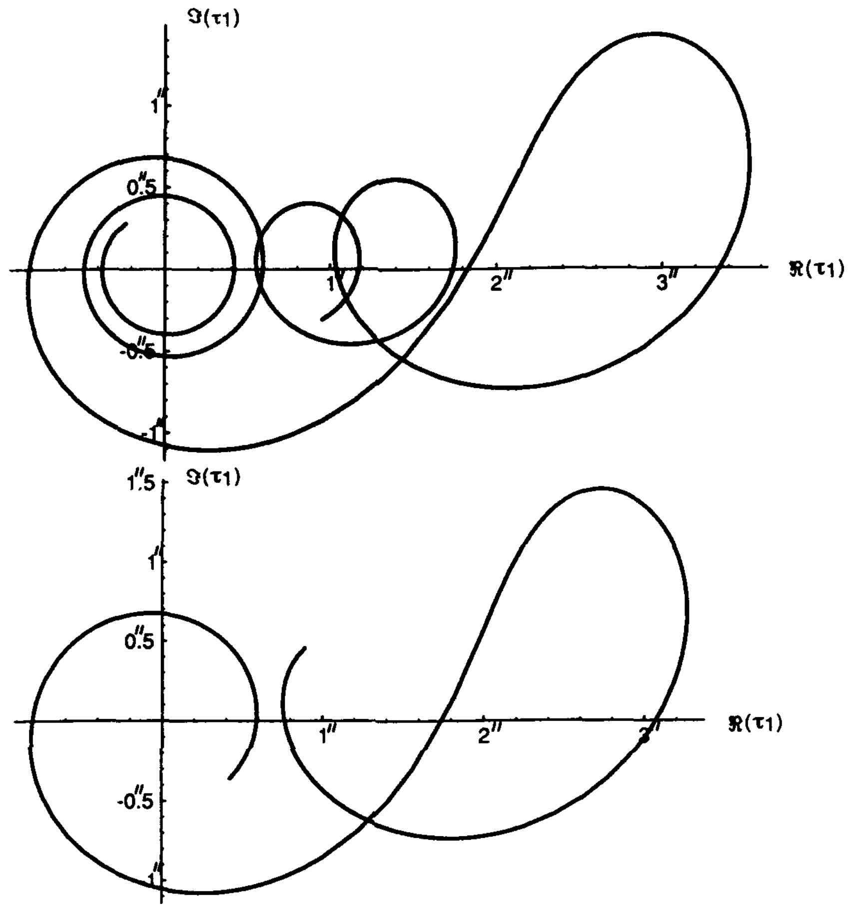

Fig. 4. The solutions to (20) for ELP-2000 Term 16331 using the Moons (top) and Migus (bottom) resonances and k given by (35.2) and (35.3). The span of the top plot is 104000 years, and that of the bottom plot is 66000 years.

curve for the Migus solution ($|\tau_1| = 1''.0$ at $\kappa = 40.1 \times 10^{-6}$) is moving upward with time. They only happen to be close to each other at the present.

Figure 5 is a plot of the libration amplitude $|\tau_1(h)|$ as a function of the dissipation parameter κ and the resonance period in sidereal months. One can see from this plot that the Figure 1 solution for the Eckhardt resonance (38.652 sidereal months) falls in a trough; the solution for the Moons resonance (38.655 sidereal months) runs along a ridge; and the solution for the Migus resonance (38.659 sidereal months) falls in a trough. A change of resonance of only 0.001 sidereal months moves a solution from a ridge to a trough. The resonance period is not known well enough to decide whether the solution runs along a ridge, in a trough, or somewhere in between.

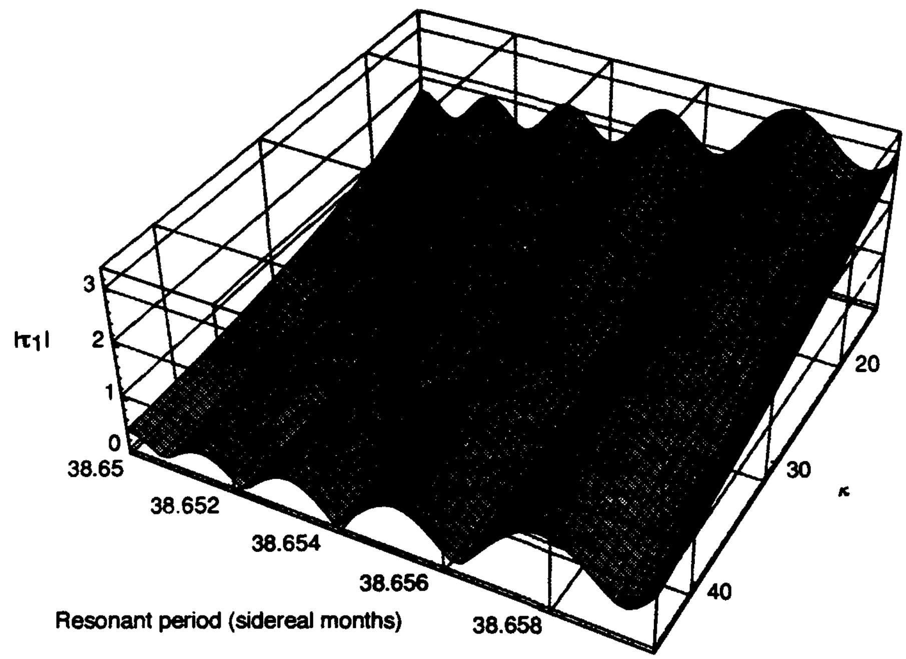

Fig. 5. The amplitude of $\tau_1(h)$ as a function of the resonance period (in sidereal months) and the dissipation parameter, κ.

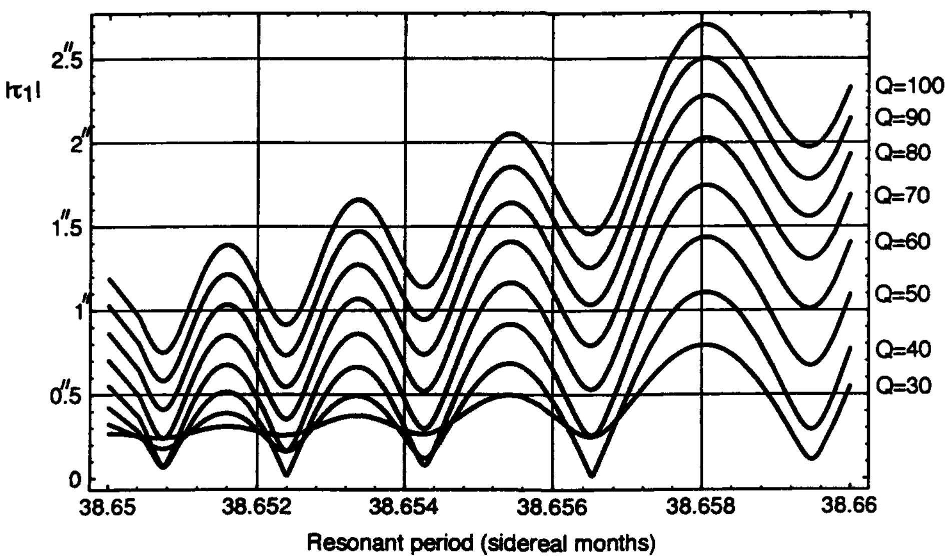

Fig. 6. Plots of $|\tau_1(h)|$ as a function of the resonance period (in sidereal months) for Q running from 30 to 100.

The steady state solution (10) for $\tau_2(h)$ has an amplitude of $0''.032$. If approximation (25) is used to estimate $|\tau_2(h)|$, the result is zero, independent of κ. If, however, (22) is used, the amplitude is $0''.032$, again independent of κ. The failure of the approximation arises from the elimination of the relatively very small term $(1-i)\kappa/(4p_k)$ in replacing (19) by (23). Because the argument of the error function is not precisely $\pi/4$ and $w(h)$ is very large, the error function has a very large real part that counters the very small $\exp(-\kappa h/2)$ attenuation term. What actually happens is that $\xi_k(\eta)$ revolves about the origin, as indicated by the solution path of (31), and the revolution rate is just the right amount to offset ω_0 to ω_2. The revolution velocity is slightly perturbed by a centripetal velocity (not acceleration) that, in effect, changes the solution phase just as the neglected $i\kappa$ term would change the solution phase in (10). The steady state solution is appropriate for ELP-2000 Term 16018, so the only term for the libration in longitude that should be modified to allow for having passed through resonance is the one due to ELP-2000 Term 16331. The steady state libration solution for this term should be replaced with a residual free libration term with unknown amplitude and phase.

6. The Q of the Moon

Seismological and astronomical techniques have been used to derive estimates of the potential-disturbance Love numbers and rigidities (or, to be more precise, their real parts) for the Earth and the Moon. The estimates are mutually consistent. This consistency unravels when it comes to estimating the imaginary parts of k and μ.

The principal effect of $\Im[k]$ in the theory of the libration of the Moon is that the node of the lunar equator on the ecliptic advances for positive $\Im[k]$. Let $\delta\Omega$ be the offset in the node; then $I\delta\Omega = \Im[k] \times 208''$ [Eckhardt, 1981]. Using (30), $\Im[k] = -\kappa/\varsigma = -0.25 \times 10^{-3}$, so, by the nominal solution, $I\delta\Omega$ decreases by $0''.052$ leading to a -45 cm $\cos F$ term in the latitude libration. Also, $Q = k\varsigma/\kappa = 100$ according to the nominal solution ($1''.0$ residual free libration and Eckhardt resonance), but if the Moons or Migus resonance is used instead, then $Q \approx 55$. Estimates of Q within the plausible range of resonances and for different residuals can be effected by using Figure 6 which is a plot of $|\tau_1(h)|$ as a function of the resonance period for $Q = 30, 40, 50, \ldots, 100$. For low Q, as at $Q = 30$, $\tau_1(h)$ oscillates about the steady state solution (10). As an example, one can see from this figure that with a resonance of 38.658 sidereal months and $Q = 40$, the residual is $|\tau_1(h)| = 1''.15$, and with the same resonance and $Q = 30$, the residual is $|\tau_1(h)| = 0''.80$. (The amplitude in the Eckhardt [1982] table is $0''.274$.) Thus a Q as low as 40, and perhaps even 30, may not be inconsistent with the observed free libration.

Williams, Newhall and Dickey [1987] estimate Q for a monthly period to be approximately 30; in effect, they detected and isolated a 1.5 m $\cos F$ term in the latitude libration. Seismic estimates of the lunar rigidity Q are depth dependent, varying from about 3000 in the region between 60 and 400 km depth to 1000 or less for depths exceeding 1100 km [Goins, Dainty and Toksoz, 1981]. The broad range of lunar Q estimates is not unlike the range of Q estimates for the Earth. Seismic rigidity (shear wave) Q's are depth dependent, varying from about 100 at the top of the mantle to about 2000 at the base of the mantle [Stacey, pp. 300-307, 1977]. Estimates of Q from polar motion data range between 30 and 600 [Munk and MacDonald, pp. 167-174, 1960; Stacey, p. 67, 1977]. The Earth has a liquid core and a more pronounced stratification than the Moon, so it is a more complicated body. It is, however, much easier to make Q-sensitive measurements of the Earth than of the Moon. That the seismic Q's of the Moon are much larger than its rotational Q's is not an inconsistency; rocks have different viscoelastic properties at $\sim$ 1 Hz than at one month or three years. The rotational Q's are roughly bracketed somewhere between 30 and 100; to some geophysicists, that's pinpoint precision.

7. Conclusion

The planetary perturbation term ELP-2000 Sequence Number 16331 [Chapront and Chapront-Touzé , 1983] modulates the lunar longitude with an amplitude of only $0''.0021$ but, because it passed through resonance 33 to 66 thousand years ago, it is the source of a $\sim 1''$ lunar libration term that has been considered a free libration. In the sense that this libration term's phase is a free parameter unrelated to other parameters of lunar motion, this is true; but in the sense that the term's amplitude is not a free parameter but, in fact, a sensitive indicator of lunar dissipation at the 38.65 sidereal month resonant period and its excitation source is known, regarding the term as a free libration is misleading by accepted standards. The definition of a free libration is hereby broadened to include this term.

Three libration theories differ in their resonance periods for the libration in longitude by less than 0.02% but the amplitude of the free libration in longitude is very sensitive to which theory is used; and, of course, it is sensitive to the imaginary part of the potential-disturbance Love number at resonance. The Moon is a fairly uniform body, and its rigidity and density are well constrained and so, therefore, is the real part of the Love number. The Q of the lunar rigidity is the dissipation parameter that, with a choice of resonance, can be estimated from the free libration amplitude. Allowing for uncertainties in the amplitude and resonance, the Moon's Q still falls within a region that is entirely reasonable by geophysical standards.

Since an astronomic excitation source for the free libration in longitude has been found, there is less justification to posit a source within the Moon, e.g., a turbulent fluid core that interacts with the mantle [Yoder, 1981]. Thus, the case for a fluid lunar core is weakened.

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